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ON THE THEORY OF β-DECAY

BY

STEFAN ROZENTAL

KØBENHAVN
EJNAR MUNKSGAARD
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1. Introduction.

In the theory developed by YUKAWA, the short range character of the nuclear forces is intimately connected with the existence of a new type of particle, the so-called meson, with a mass of about two hundred times the mass of the electron. According to this theory, the nuclear force should be due to a virtual emission and absorption of mesons by the heavy nuclear constituents which in the following will be called nucleons*. The further assumption of the possibility of processes in which mesons are similarly absorbed and emitted by the light particles (electrons, neutrinos) leads to a description of the β-decay as a complex process in which a meson, virtually created by the transition of a neutron into a proton, is immediately annihilated, emitting an electron and an antineutrino. Already in his first paper[1)], YUKAWA developed a theory on these lines; describing the meson field simply by a scalar wave-function, he found for the energy distribution of the β-rays essentially the same formula as given by the original theory of FERMI[2)].

Since the scalar theory did not give the right type of nuclear forces, a new formalism, in which the meson field is described by a vector, has been developed by several

* As regards this notation, compare reference 22.

authors[3] and the corresponding form of the theory of β-decay has been given by Yukawa and collaborators[4]. They found that the most general form of this theory, in which the expression for the interaction energy between the meson field and the light particles does not contain derivatives of wave-functions of the light particles, leads again essentially to the formula of Fermi. A distribution of the type considered by Konopinski and Uhlenbeck[5] could only be obtained by introducing an interaction explicitly involving derivatives of the neutrino wave-function. This distribution formula leads, of course, to a lifetime-energy relation for the β-radioactive elements of the same type as the original Konopinski-Uhlenbeck theory, a result which is incompatible with the experiments on ^{8}Li; these experiments seem, in fact, to be in accordance with a lifetime-energy relation of the type which follows from the Fermi formula[6]. As regards the energy distribution of the β-rays, however, the measurements on the β-spectra of different radioactive elements do not agree even for the so-called "allowed transitions" with the simple Fermi distribution, especially for the lower energies. Also the formula of Konopinski and Uhlenbeck is in obvious disagreement with recent more exact experiments.

Bethe, Hoyle and Peierls[7] have tried to eliminate this difficulty in the Fermi theory by the assumption that the measured spectra are the result of a superposition of different elementary processes of the Fermi type. Their assumption is supported by the fact that the β-decay in some cases has been found to be accompanied by a γ-radiation. According to this point of view, the shape of the β-spectrum should be connected with the frequency and intensity of the γ-rays. A real test of the assumption in the case of ^{13}N, where the positron spectrum is experimentally well-known[8], is, however, impossible at the moment since the experimental data as regards the γ-radiation obtained by different investigators[9] deviate essentially from each other, the existence of the γ-radiation even being denied by one author[10].

Apart from the discrepancies in the theory of β-decay, the vector model of the meson field leads to another difficulty in connection with the forces between the heavy nuclear particles, since the resulting expression for the interaction potential also includes a term of dipole type which is too strongly singular for small distances. Møller and Rosenfeld[11] have shown that it is possible to remedy this defect by introducing besides the vector wave-function a further pseudoscalar wave-function for the meson field. As indicated by these authors, the introduction of a pseudoscalar wave-function leads also to a generalization of the β-theory. For the interaction between the mesons and the other particles we get then new expressions which contain, just as in the vector theory, derivatives of the wave-function of the mesons. If a suitable canonical transformation[12] which separates out the static interaction between the nucleons is performed, the transformed Hamiltonian will contain a direct interaction between the nucleons and the light particles, described by an expression which, in general, also contains derivatives of the wave-functions of the light particles. Since the interaction between the nucleons and the light particles is responsible for the β-disintegration processes, we find for the energy distribution of the β-rays a formula which may deviate from the Fermi formula and, in some cases, is identical with the formula which was found by Fierz[13] in a Fermi theory starting from the most general interaction between the nucleons and the light particles.

It should be mentioned that Yukawa's theory of β-decay further supplies a connection between the lifetime of free mesons in the cosmic radiation and the lifetimes of β-radioactive elements. Taking for the universal β-decay constant of the Fermi theory the value given by Fermi[2], Yukawa found qualitative agreement between the lifetime of the mesons in cosmic radiation determined by Euler[14] and the lifetimes of the heavy β-radioactive elements. This agreement was only obtained with that expression for the energy of interaction between the mesons and the light particles which does not contain derivatives of the neutrino wave-function. In the case where such derivatives are introduced into the energy expression, the value for the lifetime of mesons turns out to be about ten thousand times too small, since it contains an extra factor of the order of the square of the ratio between the masses of the electron and the meson. It is known, however, that the value for the universal β-decay constant, as given by Fermi, is too small to account for the lifetimes of light elements, especially of ^{6}He. This means, as pointed out by Nordheim[15], that the theory of Yukawa would not give the right relation between the lifetimes of the light radioactive elements and the lifetime of the cosmic ray mesons. It can be shown[16] that this difficulty is unavoidable in any theory containing only one type of meson field and can, in principle, be removed by the introduction of a mixture of two types of meson fields. It should be noticed that recently Fermi[17] has drawn attention to the fact that the difference between the absorption of cosmic ray mesons in air and in condensed materials is due not only to the instability of the mesons but also to a polarization of the material. In the evaluation of the lifetime of cosmic ray mesons, the above effect must therefore be taken into account.

2. Survey of the theory.

Before proceeding to the main problem, we shall first give a survey of the generalized theory of the meson field including the pseudoscalar wave-function referred to above.

For the description of the neutral and the positively and negatively charged mesons it is convenient to introduce three (real) fields, the first two of which are connected with the charged mesons while the third represents the neutral mesons[18]. Each of the three types of field will be characterized by two vectors $\vec{F}$ and $\vec{U}$ and two further functions Φ and Ψ, the latter having the invariance property of a pseudoscalar. The field quantities belonging to the three different kinds of field will be distinguished by a heavy printed index, i. e. $(\vec{F}_{\mathbf{3}}, \vec{U}_{\mathbf{3}}, \Phi_{\mathbf{3}}, \Psi_{\mathbf{3}})$ represent the neutral meson field while $(\vec{F}_{\mathbf{1}}, \vec{U}_{\mathbf{1}}, \Phi_{\mathbf{1}}, \Psi_{\mathbf{1}})$ and $(F_{\mathbf{2}}, U_{\mathbf{2}}, \Phi_{\mathbf{2}}, \Psi_{\mathbf{2}})$ together describe the field of the charged mesons. It is convenient to group three corresponding quantities into a symbolic vector, *viz.*

$$\left.\begin{aligned} \vec{\boldsymbol{F}} &= (\vec{F}_{\mathbf{1}}, \vec{F}_{\mathbf{2}}, \vec{F}_{\mathbf{3}}) \\ \vec{\boldsymbol{U}} &= (\vec{U}_{\mathbf{1}}, \vec{U}_{\mathbf{2}}, \vec{U}_{\mathbf{3}}) \\ \boldsymbol{\Phi} &= (\Phi_{\mathbf{1}}, \Phi_{\mathbf{2}}, \Phi_{\mathbf{3}}) \\ \boldsymbol{\Psi} &= (\Psi_{\mathbf{1}}, \Psi_{\mathbf{2}}, \Psi_{\mathbf{3}}). \end{aligned}\right\} \quad (1)$$

The field quantities $\vec{\boldsymbol{F}}$ and $\vec{\boldsymbol{U}}$ as well as $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ are canonically conjugate, satisfying the usual commutation relations, i. e.

$$\left.\begin{aligned}\left[U_k^{\mu}(x,t),\ F_l^{\nu}(x',t)\right] &= \frac{\hbar c}{i}\,\delta(x-x')\,\delta^{\mu\nu}\,\delta_{kl}\\ \left[\Phi_k(x,t),\ \Psi_l(x',t)\right] &= \frac{\hbar c}{i}\,\delta(x-x')\,\delta_{kl}\end{aligned}\right\}\quad(2)$$

while all other pairs of field quantities commute.

We shall now write down the Hamiltonian for a system of heavy particles (protons, neutrons), light particles (electrons, neutrinos), and the meson field including the most general interaction between particles and field.

All quantities referring to the light particles will throughout be denoted by the same letters as the corresponding quantities referring to the heavy particles but with the symbol $\breve{}$ placed above. For instance, the "isotopic" spin of the nucleons is denoted by the letter $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$, where $\tau_3 = +1$ characterizes the neutron state and $\tau_3 = -1$ the proton state of the heavy particle. Accordingly, we shall use the notation $\breve{\boldsymbol{\tau}} = (\breve{\tau}_1, \breve{\tau}_2, \breve{\tau}_3)$ for the "isotopic" spin of the light particle, where $\tau_3 = .+1$ means the electron state and $\breve{\tau}_3 = -1$ the neutrino state of the light particle. Similarly, ρ, $\vec{\sigma}$ and $\breve{\rho}$, $\vec{\breve{\sigma}}$ are the usual Dirac spin variables for the nucleons and the light particles, respectively.

For later reference, we shall now list the quantities which appear in the Hamiltonian and refer to the light particles:

$$\left.\begin{aligned}\breve{\boldsymbol{N}} &= \breve{g}_1\,\psi^*\,\breve{\boldsymbol{\tau}}\,\psi\\ \vec{\breve{\boldsymbol{M}}} &= \breve{g}_1\,\psi^*\,\breve{\boldsymbol{\tau}}\,\breve{\rho}_1\,\vec{\breve{\sigma}}\,\psi\\ \vec{\breve{\boldsymbol{T}}} &= -\frac{\breve{g}_2}{\kappa}\,\psi^*\,\breve{\boldsymbol{\tau}}\,\breve{\rho}_2\,\vec{\breve{\sigma}}\,\psi\\ \vec{\breve{\boldsymbol{S}}} &= \frac{\breve{g}_2}{\kappa}\,\psi^*\,\breve{\boldsymbol{\tau}}\,\breve{\rho}_3\,\vec{\breve{\sigma}}\,\psi\\ \vec{\breve{\boldsymbol{P}}} &= \frac{\breve{f}_2}{\kappa}\,\psi^*\,\breve{\boldsymbol{\tau}}\,\vec{\sigma}\,\psi\\ \breve{\boldsymbol{Q}} &= \frac{\breve{f}_2}{\kappa}\,\psi^*\,\breve{\boldsymbol{\tau}}\,\breve{\rho}_1\,\psi\\ \breve{\boldsymbol{R}} &= \breve{f}_1\,\psi^*\,\breve{\boldsymbol{\tau}}\,\breve{\rho}_2\,\psi.\end{aligned}\right\}\quad(3)$$

Here ψ denotes the wave-function of the light particle; the $\breve{g}$'s and $\breve{f}$'s are universal constants which have the dimensions of electric charge and determine the strength of interaction between light particles and the meson field, κ is the reciprocal of the range of the nuclear forces and is connected with the mass M_m of the meson by the relation

$$\kappa = \frac{M_m c}{\hbar}. \tag{4}$$

Analogous quantities referring to the nucleons will appear in the Hamiltonian. It should be noticed that the quantities $\vec{\boldsymbol{M}}$, $\vec{\boldsymbol{T}}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ are proportional to the ratio of the velocity of the nucleons to the velocity of light and are, therefore, small compared with the quantities $\boldsymbol{N}$, $\vec{\boldsymbol{S}}$, $\vec{\boldsymbol{P}}$. We have:

$$\left.\begin{aligned}
\boldsymbol{N} &= g_1 \sum_{(i)} \overset{(i)}{\tau}\, \delta(\vec{x} - \vec{x}^{(i)}) \\
\vec{\boldsymbol{M}} &= g_1 \sum_{(i)} \overset{(i)}{\tau}\, \overset{(i)}{\rho_1}\, \overset{(i)}{\vec{\sigma}}\, \delta(\vec{x} - \vec{x}^{(i)}) \\
\vec{\boldsymbol{T}} &= -\frac{g_2}{\kappa} \sum_{(i)} \overset{(i)}{\tau}\, \overset{(i)}{\rho_2}\, \overset{(i)}{\vec{\sigma}}\, \delta(\vec{x} - \vec{x}^{(i)}) \\
\vec{\boldsymbol{S}} &= \frac{g_2}{\kappa} \sum_{(i)} \overset{(i)}{\tau}\, \overset{(i)}{\rho_3}\, \overset{(i)}{\vec{\sigma}}\, \delta(\vec{x} - \vec{x}^{(i)}) \\
\vec{\boldsymbol{P}} &= \frac{f_2}{\kappa} \sum_{(i)} \overset{(i)}{\tau}\, \overset{(i)}{\vec{\sigma}}\, \delta(\vec{x} - \vec{x}^{(i)}) \\
\boldsymbol{Q} &= \frac{f_2}{\kappa} \sum_{(i)} \overset{(i)}{\tau}\, \overset{(i)}{\rho_1}\, \delta(\vec{x} - \vec{x}^{(i)}) \\
\boldsymbol{R} &= f_1 \sum_{(i)} \overset{(i)}{\tau}\, \overset{(i)}{\rho_2}\, \delta(\vec{x} - \vec{x}^{(i)}).
\end{aligned}\right\} \tag{5}$$

In these expressions, we represent the nucleons in the configuration space, all quantities belonging to the i-th particle being denoted by the index $^{(i)}$. The constants g_1, g_2, f_1, f_2, which have the dimensions of electric charge, determine the magnitude of the nuclear forces.

As already mentioned, the vector model leads to a singular term in the static interaction energy between the nucleons; a similar term, with opposite sign, arises from the pseudoscalar part of the field. These two terms cancel each other if one puts

$$|f_2| = |g_2|. \tag{6}$$

The expression for the Hamiltonian can then be written as a sum

$$\mathcal{H} = \mathcal{H}_k + \breve{\mathcal{H}}_k + \mathcal{H}_f + H_1 + H_2 \tag{7}$$

where $\mathcal{H}_k$ and $\breve{\mathcal{H}}_k$ are the kinetic energies of the nucleons and the light particles, respectively, and $\mathcal{H}_f$ is the energy of the meson field, while the interaction energy consists of two parts: H_1 containing the constants g_1, g_2, f_2, only, and giving rise to the forces between the nuclear constituents, and H_2 which also contains the constants $\breve{f}_1$, $\breve{f}_2$, $\breve{g}_1$, $\breve{g}_2$ and is responsible for the β-disintegration. If M_N, M_P and m are the masses of the neutron, the proton and the electron, respectively, and if the mass of the neutrino is put equal to zero, the first four parts of the Hamiltonian are given by the following formulae:

$$\left.\begin{aligned}
\mathcal{H}_k &= \sum_{(i)} \left\{ \rho_1^{(i)} \vec{\sigma}^{(i)} c \vec{p}^{(i)} + \rho_3^{(i)} \left(\frac{1+\tau_3^{(i)}}{2} M_N c^2 + \frac{1-\tau_3^{(i)}}{2} M_P c^2 \right) \right\} \\
\breve{\mathcal{H}}_k &= \int \psi^* \left(\breve{\rho}_1 \vec{\breve{\sigma}} c \vec{\breve{p}} + \breve{\rho}_3 \frac{1+\breve{\tau}_3}{2} mc^2 \right) \psi \, dV \\
\mathcal{H}_f &= \frac{1}{2} \int \left\{ \vec{\boldsymbol{F}}^2 + \kappa^{-2} (\operatorname{div} \vec{\boldsymbol{F}})^2 + (\operatorname{rot} \vec{\boldsymbol{U}})^2 + \kappa^2 \vec{\boldsymbol{U}}^2 \right\} dV \\
&\quad + \frac{1}{2} \int \left\{ \boldsymbol{\Phi}^2 + (\operatorname{grad} \boldsymbol{\Psi})^2 + \kappa^2 \boldsymbol{\Psi}^2 \right\} dV
\end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} H_1 = & \int \left\{ \kappa^{-2} \left(\frac{1}{2} \boldsymbol{N}^2 - \boldsymbol{N} \operatorname{div} \vec{\boldsymbol{F}} \right) + \left(\frac{1}{2} \vec{\boldsymbol{S}}^2 + \vec{\boldsymbol{S}} \operatorname{rot} \vec{\boldsymbol{U}} \right) \right\} dV \\ & - \int \vec{\boldsymbol{P}} \operatorname{grad} \boldsymbol{\Psi} \, dV + \frac{1}{2} \int \boldsymbol{Q}^2 \, dV. \end{aligned} \right\} \quad (9)$$

The scalar products involving the symbolic isotopic vectors are analogous to products of ordinary vectors, e. g.

$$\boldsymbol{\Phi} \breve{\boldsymbol{Q}} = \sum_k \Phi_k \breve{Q}_k$$

$$\vec{\boldsymbol{F}}^2 = \sum_{\mu, k} \left(F_k^\mu \right)^2$$

$$\vec{\boldsymbol{P}} \cdot \operatorname{grad} \boldsymbol{\Psi} = \sum_{\mu, k} P_k^\mu \frac{\partial}{\partial x_\mu} \Psi_k.$$

The expression for H_2 is not uniquely determined by the requirement of relativistic invariance of the whole scheme. It is, of course, always possible to add to the Lagrangeian function invariant expressions as

$$\left.\begin{aligned} & \eta \;\; (\vec{\boldsymbol{T}} \vec{\breve{\boldsymbol{T}}} - \vec{\boldsymbol{S}} \vec{\breve{\boldsymbol{S}}}) \\ & \eta' \;\; \kappa^{-2} (\vec{\boldsymbol{M}} \vec{\breve{\boldsymbol{M}}} - \boldsymbol{N} \breve{\boldsymbol{N}}) \\ & \eta'' \, (\boldsymbol{Q} \breve{\boldsymbol{Q}} - \vec{\boldsymbol{P}} \vec{\breve{\boldsymbol{P}}}) \\ & \eta''' \, \kappa^{-2} \boldsymbol{R} \breve{\boldsymbol{R}} \end{aligned} \right\} \quad (10)$$

where η, η', η'' and η''' are arbitrary constants. The same terms would also appear in the Hamiltonian and we have then for the most general form of H_2 the expression

$$
\left.\begin{aligned}
H_2 = \int \Big\{ & \kappa^{-2}\left[\frac{1}{2}\breve{\boldsymbol{N}}^2 + \breve{\boldsymbol{N}}(-\operatorname{div}\vec{\boldsymbol{F}} + \boldsymbol{N})\right] + \\
& + \left[\frac{1}{2}\vec{\breve{\boldsymbol{S}}} + \vec{\breve{\boldsymbol{S}}}(\operatorname{rot}\vec{\boldsymbol{U}} + \vec{\boldsymbol{S}})\right] - [\vec{\breve{\boldsymbol{T}}}\vec{\boldsymbol{F}} + \vec{\breve{\boldsymbol{M}}}\vec{\boldsymbol{U}}]\Big\} dV \\
- \int & (\vec{\breve{\boldsymbol{P}}}\operatorname{grad}\boldsymbol{\psi} + \boldsymbol{\Phi}\breve{\boldsymbol{Q}} + \boldsymbol{\psi}\breve{\boldsymbol{R}} - \vec{\boldsymbol{P}}\vec{\breve{\boldsymbol{P}}})\, dV \\
+ \eta \int & (\vec{\boldsymbol{T}}\vec{\breve{\boldsymbol{T}}} - \vec{\boldsymbol{S}}\vec{\breve{\boldsymbol{S}}})\, dV + \eta'\kappa^{-2}\int(\vec{\boldsymbol{M}}\vec{\breve{\boldsymbol{M}}} - \boldsymbol{N}\breve{\boldsymbol{N}})\, dV \\
+ \eta'' \int & (\boldsymbol{Q}\breve{\boldsymbol{Q}} - \vec{\boldsymbol{P}}\vec{\breve{\boldsymbol{P}}})\, dV + \eta'''\kappa^{-2}\int \boldsymbol{R}\breve{\boldsymbol{R}}\, dV.
\end{aligned}\right\} \quad (11)
$$

The last integrals represent a direct coupling between heavy and light particles of the same type as the coupling in the original Fermi theory. In a theory like that proposed by Yukawa, where the β-process should be connected with the instability of mesons, one would not expect such direct coupling to appear in the Hamiltonian. It is seen, however, from (11), that it is impossible to choose the η's in such a way that all terms of direct coupling disappear. It is true that the terms $\boldsymbol{N}\breve{\boldsymbol{N}}$, $\vec{\boldsymbol{S}}\vec{\breve{\boldsymbol{S}}}$ and $\vec{\boldsymbol{P}}\vec{\breve{\boldsymbol{P}}}$ vanish if we choose $\eta = \eta' = \eta'' = 1$, but instead we get terms containing $\vec{\boldsymbol{M}}\vec{\breve{\boldsymbol{M}}}$, $\vec{\boldsymbol{T}}\vec{\breve{\boldsymbol{T}}}$ and $\boldsymbol{Q}\breve{\boldsymbol{Q}}$, which are again of the same type. We have, therefore, to retain the general expression without ascribing beforehand definite values to the η's. For the same reason, it is not allowed to neglect the terms containing $\vec{\boldsymbol{M}}\vec{\breve{\boldsymbol{M}}}$, $\vec{\boldsymbol{T}}\vec{\breve{\boldsymbol{T}}}$, $\boldsymbol{Q}\breve{\boldsymbol{Q}}$ and $\boldsymbol{R}\breve{\boldsymbol{R}}$ although they are of a smaller order of magnitude than the terms $\boldsymbol{N}\breve{\boldsymbol{N}}$, $\vec{\boldsymbol{S}}\vec{\breve{\boldsymbol{S}}}$ and $\vec{\boldsymbol{P}}\vec{\breve{\boldsymbol{P}}}$.

3. Derivation of the formula for the β-decay.

Since the terms H_1 and H_2 contain an interaction between the meson field and the nucleons or the light particles, the β-emission is, in this form of the theory, partly a second-order effect involving a meson in the intermediate state. It is possible, however, to perform a contact transformation[12] leading to an expression for the Hamiltonian in which the static interaction between the nucleons appears explicitly. This new form of the Hamiltonian contains, furthermore, a term of direct interaction between the nucleons and the light particles, from which the β-process can be obtained as a first-order effect in a perturbation calculation.

The unitary operator $\mathscr{S}$, which determines the contact transformation (defining any new variable A' in terms of the old variables A by the formula $A' = \mathscr{S}^{-1} A \mathscr{S}$), has the form

$$\mathscr{S} = e^{\frac{i}{\hbar c}\mathscr{K}} \tag{12}$$

with*

$$\mathscr{K} = g_1 \sum_{(i)} \boldsymbol{\tau}^{(i)} \int \vec{f}^{(i)} \vec{\boldsymbol{U}}\, dV + \frac{g_2}{\kappa} \sum_{(i)} \boldsymbol{\tau}^{(i)} \vec{\sigma}^{(i)} \int \left[(\vec{f}^{(i)} \wedge \vec{\boldsymbol{F}}) + \vec{f}^{(i)} \boldsymbol{\Phi} \right] dV \tag{13}$$

where

$$\left. \begin{aligned} \vec{f}^{(i)}(x) &= -\operatorname{grad} \frac{e^{-\kappa r_i}}{4\pi r_i} \\ r_i &= \left| \vec{x} - \vec{x}^{(i)} \right|. \end{aligned} \right\} \tag{14}$$

Since the old variables do not appear again in the following, we shall from now on omit the prime in the symbols for the new variables.

Apart from the static interaction in the Hamiltonian expressed as a function of the transformed variables we shall now only retain such interaction terms which are of importance for the β-disintegration, and we find in this way the expression

* The symbol $\wedge$ between two vectors denotes their vector product.

$$\mathcal{T} = \mathcal{H}_k + \breve{\mathcal{H}}_k + \mathcal{H}_f + H_{\text{stat}} + H_\beta \tag{15}$$

where $\mathcal{H}_k$, $\breve{\mathcal{H}}_k$ and $\mathcal{H}_f$ are given by (8), and*

$$\left.\begin{aligned} H_{\text{stat}} &= \frac{1}{2}\sum_{i,k} (\overset{(i)}{\tau}\overset{(k)}{\tau}) \left[g_1^2 + g_2^2 (\vec{\sigma}^{(i)}\vec{\sigma}^{(k)})\right] \frac{e^{-\kappa r_{ik}}}{4\pi r_{ik}} \\ r_{ik} &= \left|\vec{x}^{(i)} - \vec{x}^{(k)}\right| \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} H_\beta = \sum_{(i)} \Big\{ & g_1 \int \overset{(i)}{\tau}\, \breve{\boldsymbol{N}} \frac{e^{-\kappa r_i}}{4\pi r_i} dV + g_2 \kappa \int \overset{(i)}{\tau}\, \vec{\breve{\boldsymbol{S}}}\, \overset{(i)}{\rho}_3 \vec{\sigma}^{(i)} \frac{e^{-\kappa r_i}}{4\pi r_i} dV - \\ & - \frac{f_2}{\kappa} \int \overset{(i)}{\tau}\, \vec{\sigma}^{(i)} \frac{e^{-\kappa r_i}}{4\pi r_i} \operatorname{grad} \breve{\boldsymbol{R}}\, dV \Big\} \\ + \sum_{(i)} \Big\{ & -\eta \frac{g_2}{\kappa} \overset{(i)}{\tau}\, \vec{\sigma}^{(i)} \left[\overset{(i)}{\rho}_2 \vec{\breve{\boldsymbol{T}}}(\overset{(i)}{x}) + \overset{(i)}{\rho}_3 \vec{\breve{\boldsymbol{S}}}(\overset{(i)}{x})\right] + \\ & + \eta' \frac{g_1}{\kappa^2} \overset{(i)}{\tau} \left[\overset{(i)}{\rho}_1 \vec{\sigma}^{(i)} \vec{\breve{\boldsymbol{M}}}(\overset{(i)}{x}) - \breve{\boldsymbol{N}}(\overset{(i)}{x})\right] + \\ & + \frac{f_2}{\kappa} \overset{(i)}{\tau} \left[\eta'' \overset{(i)}{\rho}_1 \breve{\boldsymbol{Q}}(\overset{(i)}{x}) + (1-\eta'') \vec{\sigma}^{(i)} \vec{\breve{\boldsymbol{P}}}(\overset{(i)}{x})\right] + \\ & + \frac{f_1}{\kappa^2} \eta''' \overset{(i)}{\tau}\, \overset{(i)}{\rho}_2 \breve{\boldsymbol{R}}(\overset{(i)}{x}) \Big\} \\ - \sum_{(i)} \Big\{ & g_1 \int \overset{(i)}{\tau}\, \vec{\breve{\boldsymbol{T}}} \vec{f}^{(i)} dV - \frac{g_2}{\kappa} \int \overset{(i)}{\tau}\, \vec{\breve{\boldsymbol{M}}} (\vec{\sigma}^{(i)} \wedge \vec{f}^{(i)})\, dV - \\ & - \frac{g_2}{\kappa} \int \overset{(i)}{\tau}\, \vec{\breve{\boldsymbol{S}}} (\vec{\sigma}^{(i)} \operatorname{grad}) \vec{f}^{(i)} dV - \\ & - \frac{f_2}{\kappa} \int \overset{(i)}{\tau}\, \vec{\breve{\boldsymbol{P}}} \operatorname{grad} \operatorname{div} \vec{\sigma}^{(i)} \frac{e^{-\kappa r_i}}{4\pi r_i} dV \Big\}. \end{aligned}\right\} \tag{17}$$

As the β-disintegration consists of a transformation of a neutron into a proton with a simultaneous emission of an electron and an antineutrino, we now ask for the probability

* We have, already at this point, put $\overset{(i)}{\rho}_3 = 1$ in H_{stat}.

of a process in which a neutrino in a negative energy state disappears and an electron is created in a positive energy state while a neutron in the nucleus changes into a proton.

If the initial and final states of the nucleus are denoted by the letters n_0 and n, and the states of the electron and neutrino involved are described by the eigenfunctions φ_s and φ_σ, the probability per unit time for such a process to happen is equal to

$$\frac{2\pi}{\hbar}\,\delta(E_{n_0}-E_n+E_\sigma-E_s)\,|(n,s\,|\,H_\beta\,|\,\eta_0,\sigma)|^2 \tag{18}$$

where E_{n_0}, E_n, $E_\sigma\,(<0)$ and E_s are the energies of the corresponding states.

Using (6), it is easily seen by partial integration that the matrix elements of the four last integrals in (17) are small compared with the matrix elements of the three first integrals* since they will contain an extra factor of the order of the ratio between the momentum of the electron (or the neutrino) and $\kappa\hbar$, which again for ordinary β-processes does not exceed the order of magnitude $\frac{m}{M_m} \backsim 10^{-2}$. Retaining the other integrals in (17), using the definitions (3), and putting in the integrals approximately

* The partial integration of the last integral in (17) yields, furthermore, a double integral extended over a small surface around the point $x^{(i)}$. The value of this integral is

$$\mu\frac{f_2}{\kappa}\boldsymbol{\tau}^{(i)}\vec{\sigma}^{(i)}\vec{\boldsymbol{P}}(x^{(i)}) \tag{19}$$

where the constant μ depends on the shape of the surface chosen $\left(\text{e. g. } \mu=\frac{4\pi}{3} \text{ for a sphere}\right)$. Terms of this type occur already in the expression (17) and the appearance of (19) can be accounted for by changing the coefficient $(1-\eta'')$ into $(1-\eta''-\mu)$ so that the general character of the final result will not be affected by this change.

$$\frac{e^{-\kappa r_i}}{4\pi r_i} = \frac{1}{\kappa^2}\,\delta\,(\vec{x} - \vec{x}^{(i)}) \tag{20}$$

we get for (18)

$$\frac{2\pi}{\hbar}\,\delta\,(E_{n_0} - E_n + E_\sigma - E_s)\left|(n\left|H_\beta^{s\sigma}\right|n_0)\right|^2 \tag{21}$$

where

$$\left.\begin{aligned} H_\beta^{s\sigma} = \sum_{(i)} \frac{2}{\kappa^2} Q^{(i)} \Big\{ & g_1 \breve{g}_1 \varphi_s^\star \varphi_\sigma + g_2 \breve{g}_2 \rho_3^{(i)} \vec{\sigma}^{(i)} \varphi_s^\star \breve{\rho}_3 \vec{\breve{\sigma}} \varphi_\sigma - \frac{f_2 \breve{f}_1}{\kappa} \vec{\sigma}^{(i)} \operatorname{grad} \varphi_s^\star \breve{\rho}_2 \varphi_\sigma \\ & + \eta\, g_2 \breve{g}_2 \rho_2^{(i)} \vec{\sigma}^{(i)} \varphi_s^\star \breve{\rho}_2 \vec{\breve{\sigma}} \varphi_\sigma - \eta\, g_2 \breve{g}_2 \rho_3^{(i)} \vec{\sigma}^{(i)} \varphi_s^\star \breve{\rho}_3 \vec{\breve{\sigma}} \varphi_\sigma \\ & + \eta'\, g_1 \breve{g}_1 \rho_1^{(i)} \vec{\sigma}^{(i)} \varphi_s^\star \breve{\rho}_1 \vec{\breve{\sigma}} \varphi_\sigma - \eta'\, g_1 \breve{g}_1 \varphi_s^\star \varphi_\sigma \\ & + \eta''\, f_2 \breve{f}_2 \rho_1^{(i)} \varphi_s^\star \breve{\rho}_1 \varphi_\sigma + (1-\eta'')\, f_2 \breve{f}_2 \vec{\sigma}^{(i)} \varphi_s^\star \vec{\breve{\sigma}} \varphi_\sigma \\ & + \eta'''\, f_1 \breve{f}_1 \rho_2^{(i)} \varphi_s^\star \breve{\rho}_2 \varphi_\sigma \Big\} \end{aligned}\right\} \tag{22}$$

and

$$Q^{(i)} = \frac{\tau_1^{(i)} - i\,\tau_2^{(i)}}{2}$$

is an operator transforming the i'th nucleon from the neutron state into the proton state. In formula (22), the functions $\varphi_s^\star$ and φ_σ are taken on the i'th nucleon's place. We shall now introduce the following abbrevations:

$$\left.\begin{aligned} A &= \sum_{i=1}^{N} \int \Psi_n^\star Q^{(i)} \Psi_{n_0}\, dx^{(1)} \cdots dx^{(i-1)}\, dx^{(i+1)} \cdots dx^{(N)} \\ \vec{B} &= \sum_{i=1}^{N} \int \Psi_n^\star Q^{(i)} \vec{\sigma}^{(i)} \Psi_{n_0}\, dx^{(1)} \cdots dx^{(i-1)}\, dx^{(i+1)} \cdots dx^{(N)} \\ \vec{C} &= \sum_{i=1}^{N} \int \Psi_n^\star Q^{(i)} \rho_1^{(i)} \vec{\sigma}^{(i)} \Psi_{n_0}\, dx^{(1)} \cdots dx^{(i-1)}\, dx^{(i+1)} \cdots dx^{(N)} \end{aligned}\right| \tag{23}$$

$$\left.\begin{aligned}
\vec{D} &= \sum_{i=1}^{N} \int \Psi_n^\star Q^{(i)} \rho_2^{(i)} \vec{\sigma}^{(i)} \Psi_{n_0} \, dx^{(1)} \cdots dx^{(i-1)} dx^{(i+1)} \cdots dx^{(N)} \\
\vec{E} &= \sum_{i=1}^{N} \int \Psi_n^\star Q^{(i)} \rho_3^{(i)} \vec{\sigma}^{(i)} \Psi_{n_0} \, dx^{(1)} \cdots dx^{(i-1)} dx^{(i+1)} \cdots dx^{(N)} \\
F &= \sum_{i=1}^{N} \int \Psi_n^\star Q^{(i)} \rho_1^{(i)} \Psi_{n_0} \, dx^{(1)} \cdots dx^{(i-1)} dx^{(i+1)} \cdots dx^{(N)} \\
G &= \sum_{i=1}^{N} \int \Psi_n^\star Q^{(i)} \rho_2^{(i)} \Psi_{n_0} \, dx^{(1)} \cdots dx^{(i-1)} dx^{(i+1)} \cdots dx^{(N)}
\end{aligned}\right\} \quad (23)$$

where Ψ denotes the wave-function of the nucleons and N is the number of nucleons in the nucleus. The operators $\rho_1^{(i)}$ and $\rho_2^{(i)}$ are proportional to the ratio $\frac{v}{c}$ between the velocity of the i'th nucleon and the velocity of light. This means that $\vec{C}$, $\vec{D}$, F and G are small compared with $\vec{B}$ and A, respectively, while $\vec{E} = \vec{B}$, the operator $\rho_3^{(i)}$ differing from unity only by terms of second order in $\frac{v}{c}$. This fact will be of importance for the final discussion of the distribution formula.

With the above notations we get*

$$\left.\begin{aligned}
&\left(n \left| \dot{H}_\beta^{s\sigma} \right| n_0\right) = \\
&= \frac{2}{\kappa^2}\Big\{(1-\eta')\, g_1 \breve{g}_1 \int A(x)\, \varphi_s^\star(x)\, \varphi_\sigma(x)\, dx + \eta' g_1 \breve{g}_1 \int \vec{C}(x)\, \varphi_s^\star(x)\, \breve{\rho}_1 \vec{\breve{\sigma}} \varphi_\sigma(x)\, dx \\
&+ (1-\eta)\, g_2 \breve{g}_2 \int \vec{E}(x)\, \varphi_s^\star(x)\, \breve{\rho}_3 \vec{\breve{\sigma}} \varphi_\sigma(x)\, dx + \eta\, g_2 \breve{g}_2 \int \vec{D}\, \varphi_s^\star(x)\, \breve{\rho}_2 \vec{\breve{\sigma}} \varphi_\sigma(x)\, dx \\
&+ \eta'' f_2 \breve{f}_2 \int F(x)\, \varphi_s^\star(x)\, \breve{\rho}_1 \varphi_\sigma(x)\, dx + (1-\eta'')\, f_2 \breve{f}_2 \int \vec{B}(x)\, \varphi_s^\star(x)\, \vec{\breve{\sigma}} \varphi_\sigma(x)\, dx \\
&+ \eta''' f_1 \breve{f}_1 \int G(x)\, \varphi_s^\star(x)\, \breve{\rho}_2 \varphi_\sigma(x)\, dx \\
&- \frac{f_2 \breve{f}_1}{\kappa} \int \vec{B}(x) \left[\operatorname{grad} \varphi_s^\star(x) \cdot \breve{\rho}_2 \varphi_\sigma(x) + \varphi_s^\star(x)\, \breve{\rho}_2 \cdot \operatorname{grad} \varphi_\sigma(x)\right] dx \Big\}.
\end{aligned}\right\} \quad (24)$$

* In the following, x and x' denote all the spacial coordinates of the respective points.

We have now to insert the expressions for the wave-functions of the light particles. The neutrino can be represented by a plane wave of the form

$$\varphi_\sigma = a_\sigma e^{\frac{i}{\hbar}\vec{p}_\sigma \vec{x}}. \tag{25}$$

For φ_s we have to insert the wave-function of an electron in a field which outside the nucleus is a Coulomb field and inside the nucleus has a form suitably chosen to represent a mean value of the electric potential of the protons. Just as in the theory of Fermi, we may then assume that the radial part of the wave-function φ_s and its first derivatives do not vary appreciably inside a region of the extension of the nucleus. We can, thus, in (24) replace the radial part of the function φ_s by a constant equal to its value on the boundary of the nucleus. This value again does not differ very much from the corresponding value of the solution in a pure Coulomb field.

An examination of the exact wave-function of a Dirac electron in a Coulomb field shows then that, while we for light elements may replace the exact wave-function by that of a free electron, such a procedure is *not* allowed for the derivatives. Since the expression (24) also contains derivatives of the wave-function φ_s we are obliged, even for light elements, to use the exact solutions of the Dirac equation for an electron moving in a Coulomb field.

To get the probability of emission of an electron with an energy between E and $E + dE$, we have to sum (21) over all neutrino states σ and all those electron states for which the energy lies in the interval $(E, E + dE)$. In order to sum over all neutrino states, we have first of all to take the sum over the two different spin states (belonging to the same momentum and energy) of the neutrino. This summation is easily performed by using the method of CASIMIR[19]. Second, we integrate over all directions of the momentum of the neutrino and, finally, over all energies of the neutrino. Due to the δ-function in (21), the result of the last integration is that we have simply to put

$$-E_\sigma = E_{n_0} - E_n - E_s = W - E_s \tag{26}$$

where W is the energy supplied by the nucleus in the β-process.

We insert the wave-function (25) into (24) and take into account that the wave-length of the neutrino is large compared with the nuclear radius, so that the exponential factor can be put equal to 1 inside the nucleus. We get thus

$$\left.\begin{aligned}
&\left(n \middle| H_\beta^{s\sigma} \middle| n_0\right) = \\
&= \frac{2}{\kappa^2}\Big\{(1-\eta')\, g_1 \breve{g}_1 \int A(x)\, \varphi_s^\star(x)\, dx + \eta'\, g_1 \breve{g}_1 \int \vec{C}(x)\, \varphi_s^\star(x)\, \breve{\rho}_1 \vec{\breve{\sigma}}\, dx \\
&+ (1-\eta)\, g_2 \breve{g}_2 \int \vec{E}(x)\, \varphi_s^\star(x)\, \breve{\rho}_3 \vec{\breve{\sigma}}\, dx + \eta\, g_2 \breve{g}_2 \int \vec{D}(x)\, \varphi_s^\star(x)\, \breve{\rho}_2 \vec{\breve{\sigma}}\, dx \\
&+ \eta''\, f_2 \breve{f}_2 \int F(x)\, \varphi_s^\star(x)\, \breve{\rho}_1\, dx + (1-\eta'')\, f_2 \breve{f}_2 \int \vec{B}(x)\, \varphi_s^\star(x)\, \vec{\breve{\sigma}}\, dx \\
&+ \eta'''\, f_1 \breve{f}_1 \int G(x)\, \varphi_s^\star(x)\, \breve{\rho}_2\, dx \\
&- \frac{f_2 \breve{f}_1}{\kappa} \int \vec{B}(x)\, \mathrm{grad}\, \varphi_s^\star(x)\, \breve{\rho}_2\, dx \\
&- \frac{f_2 \breve{f}_1}{\kappa}\, \frac{i}{\hbar}\, \vec{p}_\sigma \int \vec{B}(x)\, \varphi_s^\star(x)\, \breve{\rho}_2\, dx \Big\}\, a_\sigma = S a_\sigma.
\end{aligned}\right\} \tag{27}$$

The first of the integrals in this formula is of just the same type as the matrix element appearing in the theory of FERMI.

The neutrino wave-function is normalized in the usual way inside a large cube of the volume V by putting

$$a_\sigma = \frac{u_\sigma}{\sqrt{V}} \tag{28}$$

with

$$u_\sigma^* u_\sigma = 1. \tag{29}$$

Using the above expression for $\left(n \left| H_\beta^{s\sigma} \right| n_0\right)$ we find for (21) a formula which contains products of an integral appearing in (27) with the complex conjugate of the same or another of these integrals, e. g.

$$\left.\begin{aligned} &\int A(x)\,\varphi_s^*(x)\,dx \cdot \int \varphi_s(x)\,A^*(x)\,dx = \\ &= \iint A(x')\,\varphi_s^*(x')\,\varphi_s(x)\,A^*(x)\,dx'\,dx \end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned} &\int \vec{C}(x)\,\varphi_s^*(x)\,\breve{\rho}_1\,\vec{\breve{\sigma}}\,dx \cdot \int \vec{\breve{\sigma}}\,\breve{\rho}_2\,\varphi_s(x)\,\vec{D}^*(x)\,dx = \\ &= i \iint \vec{C}(x')\,\varphi_s^*(x')\,\breve{\rho}_3\,\varphi_s(x)\,\vec{D}^*(x)\,dx'\,dx \\ &+ \iint \left[\vec{C}(x')\,\varphi_s^*(x') \wedge \vec{\breve{\sigma}}\,\breve{\rho}_3\right] \varphi_s(x)\,\vec{D}^*(x)\,dx'\,dx. \end{aligned}\right\} \tag{31}$$

The quantity obtained has now to be summed over all neutrino states belonging to the energy E_σ. According to the method of Casimir[19], which makes use of the relation (29), this sum is equal to

$$\sum S \frac{E_\sigma + H_\sigma}{2E_\sigma} S^*$$

where H_σ denotes the energy operator for the neutrino, and the sum is extended over all directions of the neutrino momentum $\vec{p}_\sigma$. The performance of this rather troublesome calculation leads to an expression U containing terms like

$$\left.\begin{aligned}
&\iint A(x')\,\varphi_s^\star(x')\,\varphi_s(x)\,A^\star(x)\,dx'\,dx \\
&\iint \vec{B}(x')\,\varphi_s^\star(x')\,\breve{\rho}_3\,\varphi_s(x)\,\vec{E}(x)\,dx'\,dx + c.\,c. \\
&\iint \vec{E}(x')\wedge\varphi_s^\star(x')\,\rho_2\,\vec{\breve{\sigma}}\cdot\varphi_s(x)\,\vec{C}(x)\,dx'\,dx - c.\,c. \\
&\iint \left[\vec{E}(x')\,\varphi_s(x')\,\rho_1\,\vec{\breve{\sigma}}\right]\left[\operatorname{grad}\varphi_s(x)\cdot\vec{B}(x)\right]dx'\,dx - c.\,c. \\
&\iint \left[\vec{B}(x')\operatorname{grad}\varphi_s^\star(x')\right]\left[\operatorname{grad}\varphi_s(x)\cdot\vec{B}(x)\right]dx'\,dx.
\end{aligned}\right\} \quad (32)$$

The probability of emission of an electron with energy between E_s and $E_s + dE_s$ is then, remembering (26), given by

$$P(E_s)\,dE_s = \frac{2\pi}{\hbar}\,\frac{1}{h^3 c^3}\,(W - E_s)^2\,dE_s \sum_s U, \qquad (33)$$

the summation being extended over all states of the electron belonging to the same energy E_s. In order to evaluate this sum, it will be necessary to find sums of the type

$$\sum_s \varphi_s^\star(x')\,O_{u,v}\,\varphi_s(x) \qquad (34)$$

with

$$O_{u,v} = \vec{\breve{\sigma}}^{\,u}\,\breve{\rho}_v \qquad (u = 0,1;\ v = 0,1,2,3;\ \breve{\rho}_0 = 1).$$

As already mentioned, we have to use the exact solution of the Dirac equation for the electron and to put the radial part equal to its value on the boundary of the nucleus. We shall use the solution in the form given by Rose[20], who denotes the radial part of the first and last two components of the four-component wave-function by f_κ and g_κ, respectively. The quantity κ is intimately connected with the total angular momentum quantum number j and it can take on all values except 0. The angular part of the wave-function depends upon κ and, furthermore, upon a magnetic quantum number m_κ, which can take on half-integral values depending on κ. We can thus write the sum (34) as

$$\sum_{\substack{|\kappa|=1\\ \kappa \gtrless 0}}^{\infty} \sum_{m_\kappa} \overset{*}{\varphi}_{\kappa, m_\kappa}(x')\, O_{u,v}\, \varphi_{\kappa, m_\kappa}(x) = \sum_{\substack{|\kappa|=1\\ \kappa \gtrless 0}}^{\infty} X_{u,v}^{(\kappa)}. \tag{35}$$

Here, $\varphi_{\kappa, m_\kappa}$ denotes the four-component wave-function of the electron, and the inner sum is extended to all values of m_κ which belong to a given κ.

It is found that the quantities $X_{u,v}^{(\kappa)}$ can be expressed by help of the radial parts f_κ and g_κ of the wave-functions and the unit radial vector

$$\vec{n} = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{r}.$$

For example, we get for $\kappa = -1$ the following expressions

$$\left.\begin{aligned}
X_{0,0}^{(-1)} &= 2\left[\vec{n}(x)\cdot\vec{n}(x')\right] f_{-1}(r') f_{-1}(r) + 2 g_{-1}(r') g_{-1}(r)\\
X_{0,1}^{(-1)} &= 0\\
X_{0,2}^{(-1)} &= 0\\
X_{0,3}^{(-1)} &= 2\left[\vec{n}(x)\cdot\vec{n}(x')\right] f_{-1}(r') f_{-1}(r) - 2 g_{-1}(r') g_{-1}(r)\\
\vec{X}_{1,0}^{(-1)} &= 2i\left[\vec{n}(x)\wedge\vec{n}(x')\right] f_{-1}(r') f_{-1}(r)\\
\vec{X}_{1,1}^{(-1)} &= 2i\,\vec{n}(x)\, g_{-1}(r') f_{-1}(r) - 2i\,\vec{n}(x') f_{-1}(r') g_{-1}(r)\\
\vec{X}_{1,2}^{(-1)} &= -2\,\vec{n}(x)\, g_{-1}(r') f_{-1}(r) - 2\,\vec{n}(x') f_{-1}(r') g_{-1}(r)\\
\vec{X}_{1,3}^{(-1)} &= 2i\left[\vec{n}(x)\wedge\vec{n}(x')\right] f_{-1}(r') f_{-1}(r).
\end{aligned}\right\} \tag{36}$$

Also the terms in U which contain derivatives of the electron wave-function can easily be calculated. As it will be seen, the terms of the sum (35) with $|\kappa|>2$ can be neglected in the evaluation of U, so that it will only be necessary to find besides the expressions (36) the corresponding quantities with $\kappa=+1, -2$ and $+2$.

As already mentioned, we may put the functions $f_\kappa(r)$ and $g_\kappa(r)$ equal to their values at the boundary of the nucleus, i. e. to their values, when r is given by

$$\frac{r}{\hbar/mc}=\frac{r_{\text{nucl}}}{\hbar/mc}<<1. \tag{37}$$

The functions f_κ and g_κ are given[20] by the formulae

$$\left.\begin{aligned}\begin{Bmatrix}f_\kappa\\ g_\kappa\end{Bmatrix}=\frac{\sqrt{1\mp E_s}\,(2p_s r)^{\gamma-1}\sqrt{p_s}\,e^{\pi\frac{\alpha Z E_s}{p_s}}\left|\Gamma\left(\gamma+i\frac{\alpha Z E_s}{p_s}\right)\right|}{\sqrt{\pi}\,\Gamma(2\gamma+1)}\times\\ \times\left\{e^{-ip_s r}\sqrt{-\left(\kappa-i\frac{\alpha Z}{p_s}\right)\left(\gamma+i\frac{\alpha Z E_s}{p_s}\right)}\right.\\ \left.F\left(\gamma+1+i\frac{\alpha Z E_s}{p_s},\ 2\gamma+1;\ 2ip_s r\right)\mp c.c.\right\}\end{aligned}\right\} \tag{38}$$

where $c.\,c.$ denotes the complex conjugate, F— the confluent hypergeometric function, and

$$\gamma=\sqrt{\kappa^2-\alpha^2 Z^2},$$

all quantities being expressed in atomic units.

For light elements and for the whole β-spectrum except the very lowest part of it, we can assume that

$$Z\alpha<<1 \quad\text{and}\quad \frac{Z\alpha E_s}{p_s}<<1. \tag{39}$$

Expanding the expressions (38) into series in r, putting $Z\alpha = \frac{Z\alpha E_s}{p_s} = 0$ and neglecting terms of higher order in r, we find that for negative κ the function f_κ is small of the first order in r compared with g_κ, while for positive κ it is the function g_κ which is small compared with f_κ. Furthermore, it is seen that the order of magnitude in r decreases with increasing $|\kappa|$. We have

$$\left.\begin{aligned} f_\kappa &\backsim r g_\kappa \\ f_{\kappa-1} &\backsim r f_\kappa \\ g_{\kappa-1} &\backsim r g_\kappa \end{aligned}\right\} \kappa < 0 \qquad \left.\begin{aligned} g_\kappa &\backsim r f_\kappa \\ g_{\kappa+1} &\backsim r g_\kappa \\ f_{\kappa+1} &\backsim r f_\kappa \end{aligned}\right\} \kappa > 0. \tag{40}$$

These relations show that the sum (35) can, in general, be restricted to the terms with $\kappa = \mp 1$, since the terms with higher $|\kappa|$ will be much smaller. Only in the quantities like $\sum\limits_\kappa \vec{X}_{1,0}^{(\kappa)}$ and $\sum\limits_\kappa \vec{X}_{1,3}^{(\kappa)}$, where the first term with $|\kappa| = 1$ is expressed by the "small" radial functions only, it is necessary to add one second term with $|\kappa| = 2$ which may happen to be of the same order of magnitude. The calculations show, in fact, that

$$\begin{aligned} \vec{X}_{1,0}^{(-2)} &= 6i\left[\vec{n}(x)\cdot\vec{n}(x')\right]\left[\vec{n}(x)\wedge\vec{n}(x')\right] f_{-2}(r') f_{-2}(r) - \\ &\quad - 2i\left[\vec{n}(x)\wedge\vec{n}(x')\right] g_{-2}(r') g_{-2}(r) \\ \vec{X}_{1,3}^{(-2)} &= 6i\left[\vec{n}(x)\cdot\vec{n}(x')\right]\left[\vec{n}(x)\wedge\vec{n}(x')\right] f_{-2}(r') f_{-2}(r) + \\ &\quad + 2i\left[\vec{n}(x)\wedge\vec{n}(x')\right] g_{-2}(r') g_{-2}(r), \end{aligned}$$

the first part of these quantities being negligible and the second being of the same order of magnitude as $\vec{X}_{1,0}^{(-1)}$ and $\vec{X}_{1,3}^{(-1)}$, respectively.

Also in all those terms in U which contain derivatives of the electron wave-function and, consequently, the derivatives $\frac{df_\kappa}{dr}$ and $\frac{dg_\kappa}{dr}$, the radial functions corresponding to $|\kappa| = 2$ cannot be neglected. These derivatives satisfy the differential equations

$$\left.\begin{aligned} \frac{df_\kappa}{dr} &= -\left(E_s - 1 + \frac{Z\alpha}{r}\right) g_\kappa - (1-\kappa)\frac{f_\kappa}{r} \\ \frac{dg_\kappa}{dr} &= \left(E_s + 1 + \frac{Z\alpha}{r}\right) f_\kappa - (1+\kappa)\frac{g_\kappa}{r}. \end{aligned}\right\} \quad (41)$$

It is easily seen that the order of magnitude of these derivatives decreases with increasing $|\kappa|$, with the only exception of the step from $|\kappa| = 1$ to $|\kappa| = 2$, since the coefficients $(1-\kappa)$ and $(1+\kappa)$ just vanish for $\kappa = +1$ and $\kappa = -1$, respectively.

All necessary quantities $X_{u,v}^{(\kappa)}$ being calculated, we insert them into the formula for U and put the different radial functions equal to their values at the boundary of the nucleus. These constant values can be taken out of the integrals (32). Under the integral sign, there will remain two of the functions $A, B, \cdots, G$, and some combinations of the vectors $\vec{n}(x)$ and $\vec{n}(x')$.

In the course of the calculation, we have to introduce no less than 28 different types of such integrals, each of them being a functional depending on two of the functions $A, B, \cdots, G$, but the summation makes most of them disappear from the formulae. The remaining integrals are the following:

$$\left.\begin{aligned} \omega_1(W, V) &= \iint W(x')\, V^*(x)\, dx'\, dx \\ \omega_2(Y, Z) &= \iint \vec{Y}(x')\, \vec{Z}^*(x)\, dx'\, dx \end{aligned}\right\} \quad (42)$$

$$\left.\begin{aligned}
\omega_3(W, Y) &= \iint W(x')\left[\vec{n}(x) - \vec{n}(x')\right]\vec{Y}^*(x)\,dx'\,dx \\
\omega_4(W, Y) &= \iint W(x')\,\vec{n}(x)\,\vec{Y}^*(x)\,dx'\,dx \\
\omega_5(Y, Z) &= \iint \vec{Y}(x') \wedge \left[\vec{n}(x) - \vec{n}(x')\right]\vec{Z}^*(x)\,dx'\,dx \\
\omega_6(Y, Z) &= \iint \left[\vec{Y}(x')\,\vec{n}(x)\right]\left[\vec{n}(x)\,\vec{Z}^*(x)\right]dx'\,dx \\
\omega_7(Y, Z) &= \iint \vec{Y}(x')\left[\vec{n}(x) \wedge \vec{n}(x')\right]\left[\vec{n}(x)\,\vec{Z}^*(x)\right]dx'\,dx \\
\omega_8(Y, Z) &= \iint \left[\vec{Y}(x')\,\vec{n}(x')\right]\left[\vec{n}(x')\,\vec{n}(x)\right]\left[\vec{n}(x)\,\vec{Z}^*(x)\right]dx'\,dx \\
\omega_9(W, Y) &= \iint W(x')\left[\vec{n}(x')\,\vec{n}(x)\right]\left[\vec{n}(x)\,\vec{Y}(x)\right]dx'\,dx,
\end{aligned}\right\} \tag{42}$$

where W and V denote the scalar functions A, F, G, while $\vec{Y}$ and $\vec{Z}$ stand for the vectorial functions $\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$.

In the final formula, terms appearing with the same constants can be compared as regards the order of magnitude. In order to eliminate those expressions which are small, we notice that

$$\frac{mc}{\kappa\hbar} = \frac{m}{M_m} \backsim 10^{-2} \tag{43}$$

and that for the light nuclei considered here

$$\frac{r_{\text{nucl}}}{\hbar/mc} \backsim \alpha Z^{1/3} \backsim 10^{-2} \tag{44}$$

while E_s and p_s are of the order of magnitude 1. Furthermore, the functions F and G are small of the order of magnitude $\frac{v}{c} \backsim 10^{-1}$ (see p. 17) compared with A, and so are $\vec{C}$ and $\vec{D}$ compared with $\vec{B}$, while, as already mentioned,

$$\vec{E} = \vec{B}$$

apart from terms of the order $\left(\frac{v}{c}\right)^2$.

Two of the matrix elements which appear in the distribution formula are known from earlier theories, *viz.*

$$\omega_1(A, A) = \left|\int A(x)\, dx\right|^2 \tag{45}$$

and

$$\omega_2(B, B) = \left|\int \vec{B}(x)\, dx\right|^2. \tag{46}$$

The quantity (45) is just the matrix element appearing in the FERMI theory and its absolute value cannot exceed the order of magnitude 1:

$$\omega_1(A, A) \leq [\omega_1(A, A)]_{\max} \sim 1.$$

The quantity (46) which also appears in the vector theory given by YUKAWA is of the type introduced already before by GAMOW and TELLER[21] in order to account for the experimental selection rules. Since

$$\int B_x(x)\, dx, \quad \int B_y(x)\, dx, \quad \int B_z(x)\, dx$$

differ from

$$\int A(x)\, dx$$

only by the appearance of the spin components σ_x, σ_y, σ_z, respectively, whose squares are equal to 1, the quantities

$$\left|\int B_x(x)\, dx\right|^2, \quad \left|\int B_y(x)\, dx\right|^2, \quad \left|\int B_z(x)\, dx\right|^2$$

do not exceed the order of magnitude 1, i. e.

$$\omega_2(B, B) \leq [\omega_2(B, B)]_{\max} \sim 3.$$

We shall confine ourselves to the consideration of the so-called allowed transitions, which means that the quantities (45) and (46) do not vanish but nearly attain their maximal values. In some cases, which will be discussed later, the same assumption has to be made as regards the matrix elements

$$\omega_2(C,C),\quad \omega_2(D,D),\quad \omega_1(F,F),\quad \omega_1(G,G), \tag{47}$$

too.

The summation over all electron states being performed, we get with the significations indicated above the following expression for $P(E_s)\,dE_s$, where E_s, W, p_s, p_σ and r are expressed in atomic units mc^2, mc and $\frac{\hbar}{mc}$, respectively, and the value

$$p_s = \sqrt{E_s^2 - 1}$$

is inserted:

$$P(E_s)\,dE_s = \tag{48}$$

$$= \frac{2}{\pi^3}\left(\frac{mc}{\kappa\hbar}\right)^4\left(\frac{mc^2}{\hbar}\right) E_s\sqrt{E_s^2-1}\,(W-E_s)^2\,dE_s \times$$

$$\times\left\{\left\{\frac{f_1^2\breve{f}_1^2}{(\hbar c)^2}\eta'''^2\,\omega_1(G,G)\right.\right. \tag{I}$$

$$+\frac{f_2^2\breve{f}_2}{(\hbar c)^2}\left[\eta''^2\,\omega_1(F,F)+(1-\eta'')^2\,\omega_2(B,B)\right] \tag{II}$$

$$+\frac{g_1^2\,\breve{g}_1^2}{(\hbar c)^2}\left[\eta'^2\,\omega_2(C,C)+(1-\eta')^2\,\omega_1(A,A)\right] \tag{III}$$

$$+\frac{g_2^2\,\breve{g}_2^2}{(\hbar c)^2}\left[\eta^2\,\omega_2(D,D)+(1-\eta)^2\,\omega_2(E,E)\right] \tag{IV}$$

$$+\frac{f_2 g_2 \breve{f}_1 \breve{g}_2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)(1-\eta)\left[\omega_6\left(E,\frac{1}{r}B\right)+\omega_6^{*}\left(E,\frac{1}{r}B\right)\right]Z\alpha \tag{V}$$

$$+\left.\frac{f_2^2\breve{f}_1^2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)\omega_8\left(B\frac{1}{r},\ \frac{1}{r}B\right)(Z\alpha)^2\right\} \tag{VI}$$

$$-\frac{1}{E_s}\left\{\frac{f_1 f_2 \breve{f}_1 \breve{f}_2}{(\hbar c)^2} i\eta''\eta'''\left[\omega_1(F,G)-\omega_1^*(F,G)\right]\right. \quad \text{(VII)}$$

$$+\frac{f_2 g_2 \breve{f}_2 \breve{g}_2}{(\hbar c)^2}(1-\eta)(1-\eta'')\left[\omega_2(B,E)+\omega_2^*(B,E)\right] \quad \text{(VIII)}$$

$$+\frac{g_1 g_2 \breve{g}_1 \breve{g}_2}{(\hbar c)^2} i\eta\eta'\left[\omega_2(C,D)-\omega_2^*(C,D)\right] \quad \text{(IX)}$$

$$\left.+\frac{f_2^2 \breve{f}_1 \breve{f}_2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)(1-\eta'')\left[\omega_6\left(B,\frac{1}{r}B\right)+\omega_6^*\left(B,\frac{1}{r}B\right)\right]Z\alpha\right\} \quad \text{(X)}$$

$$+\frac{1}{9}\frac{p_\sigma p_s^2}{E_s}\frac{f_2 g_2 \breve{f}_1 \breve{g}_2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right) i\eta\left[\omega_5(B,D)-\omega_5^*(B,D)\right] r_{\text{nucl}} \quad \text{(XI)}$$

$$-\frac{4}{3}\frac{p_\sigma}{E_s}\frac{f_2^2 \breve{f}_1 \breve{f}_2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)(1-\eta'')\,\omega_2(B,B) \quad \text{(XII)}$$

$$+\frac{1}{3}p_\sigma\frac{f_2 g_2 \breve{f}_1 \breve{g}_2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)(1-\eta)\left[\omega_2(B,E)+\omega_2^*(B,E)\right] \quad \text{(XIII)}$$

$$+\frac{1}{3}p_s^2\frac{f_2^2 \breve{f}_1^2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)\omega_2(B,B) \quad \text{(XIV)}$$

$$+\frac{1}{3}\frac{p_s^2}{E_s}\left\{-\frac{f_2^2 \breve{f}_1 \breve{f}_2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right) i\eta''\left[\omega_3(F,B)-\omega_3^*(F,B)\right] r_{\text{nucl}}\right. \quad \text{(XV)}$$

$$-\frac{f_2 g_2 \breve{f}_1 \breve{g}_2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right) i\eta\left[\omega_7\left(D,\frac{1}{r}B\right)-\omega_7^*\left(D,\frac{1}{r}B\right)\right] r_{\text{nucl}}\,Z\alpha \quad \text{(XVI)}$$

$$+\frac{f_2 g_2 \breve{f}_1 \breve{g}_2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)(1-\eta)\left[\omega_2(B,E)+\omega_2^*(B,E)\right] \quad \text{(XVII)}$$

$$\left.+\frac{f_2^2 \breve{f}_1^2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)\left[\omega_6\left(B,\frac{1}{r}B\right)+\omega_6^*\left(B,\frac{1}{r}B\right)\right]Z\alpha\right\} \quad \text{(XVIII)}$$

$$+\frac{2}{3}p_s^2\left\{\frac{f_1 g_2 \breve{f}_1 \breve{g}_2}{(\hbar c)^2}\eta'''(1-\eta)\left[\omega_3(G,E)+\omega_3^*(G,E)\right] r_{\text{nucl}}\right. \quad \text{(XIX)}$$

$$-\frac{f_2 g_1 \breve{f}_2 \breve{g}_1}{(\hbar c)^2}\eta'(1-\eta'')\left[\omega_5(B,C)+\omega_5^*(B,C)\right] r_{\text{nucl}} \quad \text{(XX)}$$

$$\left.+\frac{1}{2}\frac{f_1 f_2 \breve{f}_1^2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)\eta'''\left[\omega_3(G,B)+\omega_3^*(G,B)\right] r_{\text{nucl}}\right\} \quad \text{(XXI)}$$

$$+\frac{1}{3}\,p_\sigma^2\,\frac{f_2^2\breve{f}_1^2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)^2\omega_2(B,B) \qquad \text{(XXII)}$$

$$+\frac{1}{3}\,p_\sigma\,\frac{f_2^2\breve{f}_1^2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)^2\left[\omega_6\left(B,\frac{1}{r}B\right)+\omega_6^\star\left(B,\frac{1}{r}B\right)\right]Z\alpha \qquad \text{(XXIII)}$$

$$-\frac{1}{3}\frac{p_s^2}{E_s}\,\frac{f_1 f_2\breve{f}_1^2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)\eta'''\left[\omega_4(G,B)+\omega_4^\star(G,B)-\right. \qquad \text{(XXIV)}$$

$$\left.-\,\omega_9\left(G,\frac{1}{r}B\right)r_{\text{nucl}}-\omega_9^\star\left(G,\frac{1}{r}B\right)r_{\text{nucl}}\right]Z\alpha$$

$$+\frac{1}{9}\frac{p_\sigma p_s^2}{E_s}\left\{\frac{f_1 f_2\breve{f}_1^2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)\eta'''\left[\omega_3(G,B)+\omega_3^\star(G,B)\right]r_{\text{nucl}}\right. \qquad \text{(XXV)}$$

$$\left.+\frac{f_2^2\breve{f}_1^2}{(\hbar c)^2}\left(\frac{mc}{\kappa\hbar}\right)^2\cdot 2\,\omega_2(B,B)\right\}\Bigg\}. \qquad \text{(XXVI)}$$

The distribution formula is supposed to hold for the emission of electrons. To get the corresponding formula for the emission of positrons, the quantities E_s and E_σ in the preceding calculations have to be replaced by $-E_s$ and $-E_\sigma$, respectively. This means, according to (26), that the final distribution formula follows from (48) in changing W into $-W$ and E_s into $-E_s$.

4. Discussion of the disintegration formula.

The first six terms of the expression (48) are of the Fermi type and it is seen that no set of the constants η, η', η'', η''' can make them disappear. The terms (V) and (VI) depend besides on the matrix elements of the element in question also on its nuclear charge. This dependence is a consequence of the fact that the Hamiltonian also contains derivatives of the electron wave-function.

In the formula (48) appear a number of universal constants as well as the constants η, η', η'', η'''. Møller[22]) has recently pointed out that the theory may be brought into an especially symmetric form in which the universal constants hitherto arbitrary are connected by certain relations. In this way, the number of constants will be decreased. In section 5, the distribution formula will be discussed under the assumption that the constants are fixed in this definite way. In the present section, the discussion is carried out for the case where no assumptions are made as to the relative magnitude of the universal constants involved. It will be seen, however, that we get the same types of energy distribution formulae as in the case discussed in section 5.

In order to compare the different terms in (48) we notice that the constants g_1 and g_2 are determined from the binding energy of the deuteron and the range of nuclear forces to be

$$\frac{g_1^2}{4\pi\hbar c} = \frac{1}{35} \qquad \frac{g_2^2}{4\pi\hbar c} = \frac{1}{15}. \tag{49}$$

They are of the same order of magnitude and so is, according to (6), also the constant f_2. We have, furthermore, the relations (43) and (44) so that, for example,

$$\omega_8\left(E, \frac{1}{r} B\right) Z\alpha \sim \omega_8(E, B) \frac{Z\alpha}{r_{\text{nucl}}} \sim \omega_8(E, B)$$

for the light elements considered here. In comparing the different terms with the terms (II), (III) and (IV) we have, in general, to compare them with their second parts, since the quantities F, $\vec{C}$, $\vec{D}$ are smaller than A and $\vec{B}$, respectively. But in the case when the corresponding constant η is equal to 1, only the first parts of (II), (III), and (IV) are different from 0, and it will thus be necessary to assume that, in these cases, one or more of the integrals (47) obtain, for allowed transitions, their respective maximum values.

As nothing is known about the relative magnitude of the constants $\breve{f}_1$, $\breve{g}_1$, $\breve{f}_2$, $\breve{g}_2$, it is convenient to distinguish two cases:

α) $\breve{f}_1$ does not exceed the order of magnitude of the constants $\breve{f}_2$ and $\breve{g}_2$,

β) $\breve{f}_1$ is much greater than $\breve{f}_2$ and $\breve{g}_2$.

An examination shows that, in the case (α), the terms (V), (X)—(XIII), (XV)—(XVII) and (XIX)—(XXVI) are small compared with the terms (I)—(IV) and (VI). The terms (VI), (XIV) and (XVIII) are also small, except in one case, *viz.* if

$$\breve{f}_1 \eta''' \text{ does not greatly exceed the order of } \breve{f}_2 \tag{50}$$

and, at the same time,

$$\text{and} \quad \left.\begin{array}{c} \eta = \eta'' = 1 \\ \eta' = 1 \quad \text{or} \quad \breve{f}_1 >> \breve{g}_1. \end{array}\right\} \tag{51}$$

Under the assumption (α) and excepting the above case, we obtain the following disintegration formula:

$$P(E) = k \cdot F(E) \left[1 + \frac{\lambda}{E}\right] \tag{52}$$

where k is a constant, E is the energy of the electron, $F(E)$ is the Fermi function

$$F(E) = E\sqrt{E^2 - 1}\,(W - E)^2$$

and λ is a constant which is seen to be of absolute value smaller than 1

$$|\lambda| < 1.$$

The formula (52) is just the same as that found by FIERZ[13)] in the most general FERMI theory.

In the case indicated by (50) and (51) and still under the assumption (α), we get, because of the supplementary terms (VI), (XIV), and (XVIII), the following formula:

$$P(E) = k'\cdot F(E)\left\{\left[1 + aZ^{4/3}\right] + b\frac{1}{E} + \frac{1}{3}c\,(E^2-1) + \frac{1}{3}\,d\,\frac{E^2-1}{E}\,Z^{2/3}\right\}. \quad (53)$$

Here, a, b, c and d are constants depending on the universal constants and certain matrix elements. The order of magnitude of the constants a and d cannot exceed that of c, which again is $\leqq 1$. Furthermore,

$$c > 0, \quad \text{and} \quad |b| < 1.$$

If, on the contrary, $\breve{f}_1$ is so great that the relation (β) holds, two cases are to be distinguished.

If

$$f_1\,\eta''' >> f_2 \quad (54)$$

we are again led to formula (52). If (54) is not true, we obtain a formula containing two further terms with the constants f and g:

$$\left.\begin{aligned} P(E) = k''\cdot F(E)\Big\{&\left[1 + aZ^{4/3} + fZ^{2/3}\right] + \\ +\frac{1}{E}\left[b + gZ^{2/3}\right] + \frac{1}{3}c\,(E^2-1) &+ \frac{1}{3}\,d\,\frac{E^2-1}{E}\,Z^{2/3}\Big\}. \end{aligned}\right\} \quad (55)$$

The β-spectrum described by this formula is of quite the same type as that given by (53).

The terms proportional to $\frac{1}{E}$ and $\frac{E^2-1}{E}$ change their sign for positron emission but, since the corresponding constant factors b and d can be of both signs, there is no asymmetry between the emission of electrons and that of positrons.

The β-ray spectrum for ^{13}N ($W = 3{,}4$) as given by formula (52) is shown in Fig. 1, where curve I represents the pure

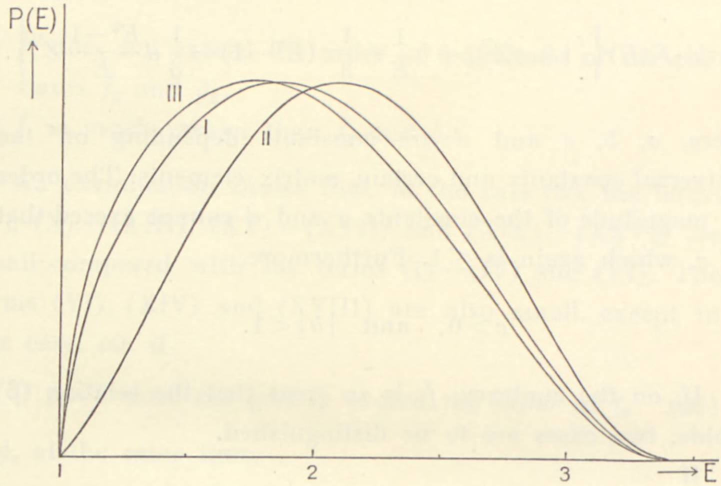

Fig. 1.

Fermi distribution ($\lambda = 0$), and the curves II and III correspond to the extreme values $\lambda = -1$ and $\lambda = +1$, respectively. The curves are normalized in such a way as to have the same height at the maximum.

The supplementary terms appearing in the expressions (53) and (55), i. e.

$$G(E) = \frac{1}{3} F(E)(E^2 - 1) \quad \text{and} \quad H(E) = \frac{1}{3} F(E) \frac{E^2 - 1}{E} \tag{56}$$

are plotted in Fig. 2 together with a pure Fermi distribution, for reference.

Since it is impossible, at the present moment, to calculate the matrix elements involved and, consequently, to estimate the values of the constants $a, b, \cdots, g$, nothing definitive can be said about the shape of the distribution curves in the different cases considered here. The formula can lead to distribution curves with a maximum lying

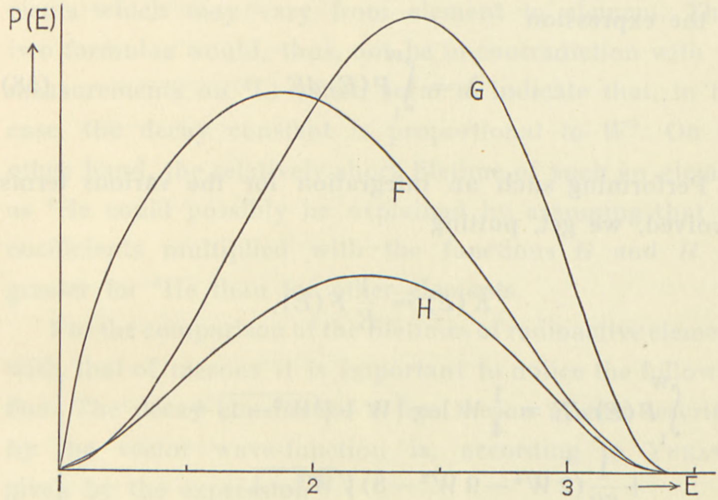

Fig. 2.

either at higher or at lower energies than that for a FERMI curve.

Among the four light elements which lie at the upper boundary of the SARGENT area,

$$^{6}\mathrm{He},\quad ^{17}\mathrm{F},\quad ^{15}\mathrm{O},\quad ^{13}\mathrm{N},\quad ^{11}\mathrm{C} \tag{57}$$

and which, therefore, may be considered as those with allowed transitions, ^{13}N is the only element for which the β-spectrum was measured with sufficient accuracy. RICHARDSON[9)] and LYMAN[9)] found a nuclear γ-radiation of about 280 kV energy; their measurements of the relative intensity of this radiation differ, however, considerably from each other. VALLEY[10)], on the other hand, could not find any radiation of this energy at all. As long as no more exact measurements are available the question of the shape of the elementary components of which the β-spectrum of ^{13}N is built up still remains open.

The decay constant of the radioactive elements is given by the expression

$$\lambda = \int_1^W P(E)\,dE. \tag{58}$$

Performing such an integration for the various terms involved, we get, putting

$$K(E) = \frac{1}{E}\,F(E),$$

$$\left.\begin{aligned}
\int_1^W F(E)\,dE &= \frac{1}{4}\,W \log\left(W + \sqrt{W^2-1}\right) + \\
&\quad + \frac{1}{60}\left(2\,W^4 - 9\,W^2 - 8\right)\sqrt{W^2-1} \\
\int_1^W G(E)\,dE &= -\frac{1}{24}\,W \log\left(W + \sqrt{W^2-1}\right) + \\
&\quad + \frac{1}{2520}\left(8\,W^6 - 38\,W^4 + 87\,W^2 + 48\right)\sqrt{W^2-1} \\
\int_1^W H(E)\,dE &= \frac{1}{24}\left(6\,W^2 + 1\right) \log\left(W + \sqrt{W^2-1}\right) + \\
&\quad + \frac{1}{360}\left(4\,W^5 + 52\,W^3 - 161\,W\right)\sqrt{W^2-1} \\
\int_1^W K(E)\,dE &= -\frac{1}{8}\left(4\,W^2 + 1\right) \log\left(W + \sqrt{W^2-1}\right) + \\
&\quad + \frac{1}{24}\left(2\,W^3 + 13\,W\right)\sqrt{W^2-1}\,.
\end{aligned}\right\} \tag{59}$$

These expressions show that, for increasing W, the different integrals increase as the 5th, 7th, 6th, and 4th power of W, respectively. Thus, if the formula (51) is valid, the decay constant is given mostly by the fifth power of W. In case the formulae (53) or (55) are valid, the life-time-energy connection depends essentially on the coefficients which may vary from element to element. These two formulae would, thus, not be in contradiction with the measurements on ^{8}Li which seem to indicate that, in this case, the decay constant is proportional to W^5. On the other hand, the relatively short lifetime of such an element as ^{6}He could possibly be explained by assuming that the coefficients multiplied with the functions G and H are greater for ^{6}He than for other elements.

For the comparison of the lifetimes of radioactive elements with that of mesons it is important to notice the following fact. The decay constant of a free meson at rest described by the vector wave-function is, according to YUKAWA, given by the expression*

$$\lambda_{\text{mes}} = \frac{M_m c^2}{4\pi\hbar}\left\{\frac{2}{3}\frac{\breve{g}_1^2}{\hbar c} + \frac{1}{3}\frac{\breve{g}_2^2}{\hbar c}\right\}. \tag{60}$$

Besides the mesons given by the vector wave-function the present theory implies the existence of other mesons originating from the pseudoscalar wave-function. Since the probability for the disintegration of these mesons[23)] is proportional, with nearly the same coefficient, to $\frac{1}{\hbar c}\left(\breve{f}_1 + \breve{f}_2 \frac{mc}{\kappa\hbar}\right)^2$ it is seen that, in the case (β), they will be much less stable than those described by a vector. The lifetime found

* In the present formulation, $\frac{g_i^2}{4\pi}$ and $\frac{\breve{g}_i^2}{4\pi}$ are equivalent to g_i^2 and $(g'\lambda_i)^2$, respectively, used in YUKAWA's paper, in analogy to the change of units in electrodynamics from HEAVISIDE units to absolute units.

from an analysis of the cosmic radiation refers to the "vector"-mesons only, since the "pseudoscalar" ones disintegrate almost at once. For the lifetimes of radioactive elements, however, both kinds of mesons are of importance.

Finally, it should be emphasized that the variation of the atomic number for the elements (57) is so small that the influence of the coefficients $Z^{2/3}$ and $Z^{4/3}$ in the formulae (53) and (55) is far less important than that of the change of the matrix elements which determine the magnitude of the coefficients a, d, f and g.

5. Discussion of the disintegration formula with fixed universal constants.

Møller[22] has shown that it is possible to unite both kinds of meson fields, i. e. the vector and the pseudoscalar field into one consistent five-dimensional tensor scheme. This description leads to an interdependence between the universal constants involved, namely

$$\left.\begin{aligned} f_1 &= g_1 \\ f_2 &= -g_2 \\ \breve{f}_1 &= \breve{g}_1 \\ \breve{f}_2 &= -\breve{g}_2. \end{aligned}\right\} \quad (61)$$

Is is, furthermore, necessary that the new scheme is invariant with respect to rotations in the five-dimensional space. The quantities (10) are invariant to Lorentz transformations only, i. e. to a certain subgroup of the group of rotations just mentioned. In the general case, the quantities (10) will be transformed into each other and the requirement of invariance leads to the relations

$$\left.\begin{aligned} \eta &= \eta'' \\ \eta' &= \eta'''. \end{aligned}\right\} \quad (62)$$

We introduce the values (61) and (62) into the distribution formula (48) and get, thus, an expression containing the constants f_1, f_2, $\breve{f}_1$, $\breve{f}_2$, η and η', only.

The discussion performed in the preceding section holds also for this special case. It will, however, be convenient to repeat the discussion in order to get more detailed informations about the coefficients appearing in the distribution formula and to show that we really obtain all the expressions for the decay probability as in the most general case. We have to distinguish three cases as regards the relative magnitude of the constants $\breve{f}_1$ and $\breve{f}_2$, the constants f_1 and f_2 being determined by (49):

$$\begin{array}{ll} (\alpha) & \breve{f}_1 << \breve{f}_2 \\ (\beta) & \breve{f}_1 \backsim \breve{f}_2 \\ (\gamma) & \breve{f}_1 >> \breve{f}_2. \end{array}$$

In case (α) we get a simple FERMI distribution

$$P(E) = k_0 \cdot F(E). \quad (63)$$

In case (β) we have to retain in (48) the terms (I), (II), (III), (IV), (VII), (VIII) and (IX). We get, thus,

$$P(E) = k_1 \cdot F(E) \left\{ a_1 + \frac{b_1}{E} \right\}. \quad (64)$$

In the special case, only, where

$$\eta = \eta' = 1$$

we have also to take the terms (VI), (XIV) and (XVIII) into consideration and we obtain the formula

$$P(E) = k_2 \cdot F(E) \left\{ a_2 + \frac{b_2}{E} + \frac{1}{3} c_2 (E^2 - 1) + \frac{1}{3} d_2 \frac{E^2 - 1}{E} \right\} \tag{65}$$

where

$$\left.\begin{aligned} a_2 = {} & f_1^2 \breve{f}_1^2 \left[\omega_1 (G, G) + \omega_2 (C, C)\right] \\ & + f_2^2 \breve{f}_2^2 \left[\omega_1 (F, F) + \omega_2 (D, D)\right] \\ & + f_2^2 \breve{f}_1 \left(\frac{mc}{\kappa \hbar}\right) \omega_8 \left(B \frac{1}{r}, \frac{1}{r} B\right) (Z\alpha)^2 \\ b_2 = {} & - f_1 f_2 \breve{f}_1 \breve{f}_2 \, i \left[\omega_1 (F, G) - \omega_1^\star (F, G) + \omega_2 (C, D) - \omega_2^\star (C, D)\right] \\ c_2 = {} & f_2^2 \breve{f}_1^2 \left(\frac{mc}{\kappa \hbar}\right) \omega_2 (B, B) \\ d_2 = {} & f_2^2 \breve{f}_1^2 \left(\frac{mc}{\kappa \hbar}\right) \left[\omega_6 \left(B, \frac{1}{r} B\right) + \omega_6^\star \left(B, \frac{1}{r} B\right)\right] Z\alpha. \end{aligned}\right\} \tag{66}$$

It is seen that $\left|\frac{b_2}{a_2}\right|$ cannot exceed 1, that c_2 and d_2 are of the same order of magnitude and that $c_2 > 0$.

In case (γ), finally, we get again the same expression (64) except if

$$\eta' = 1.$$

In this case, we obtain the formula (65) with the same values for c_2 and d_2 and with

$$\left.\begin{aligned} a_2 = {} & f_1^2 \breve{f}_1^2 \left[\omega_1 (G, G) + \omega_2 (C, C)\right] \\ & + f_2^2 \breve{f}_2^2 \left[\eta^2 \left(\omega_1 (F, F) + \omega_2 (D, D)\right) + (1 - \eta)^2 \left(\omega_2 (B, B) \, \omega_2 (E, E)\right)\right] \\ & + f_2^2 \breve{f}_1^2 \left(\frac{mc}{\kappa \hbar}\right) \omega_8 \left(B \frac{1}{r}, \frac{1}{r} B\right) (Z\alpha)^2 \\ b_2 = {} & - f_1 f_2 \breve{f}_1 \breve{f}_2 \, i \eta \left[\omega_1 (F, G) - \omega_1^\star (F, G) + \omega_2 (C, D) - \omega_2^\star (C, D)\right] \\ & - f_2^2 \breve{f}_2^2 (1 - \eta)^2 \left[\omega_2 (B, E) + \omega_2^\star (B, E)\right]. \end{aligned}\right\} \tag{67}$$

The influence of the asymmetrical term containing $\frac{1}{E}$ was shown in fig. 1. The other terms with $(E^2 - 1)$ and $\frac{E^2 - 1}{E}$ do not change the distribution curve essentially.

Fig. 3 demonstrates the influence of these additional terms. Besides a pure FERMI distribution (F) with

$$b_2 = c_2 = d_2 = 0$$

the expression (65) is plotted for the following sets of constants:

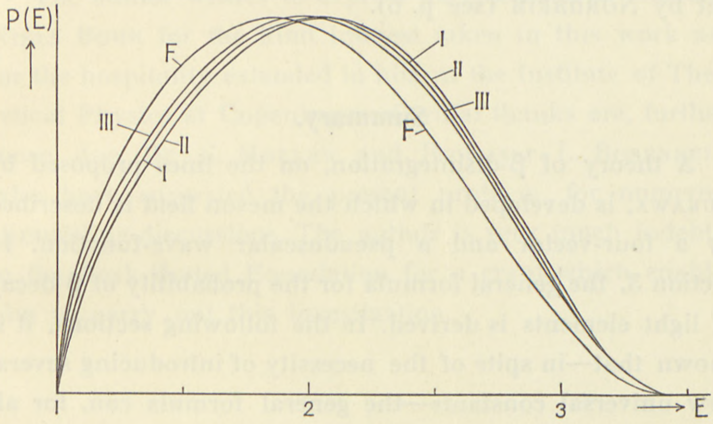

Fig. 3.

$$b_2 = 0 \qquad \frac{1}{3}\frac{c_2}{a_2} = 0.1 \qquad \frac{1}{3}\frac{d_2}{a_2} = 0 \qquad \text{(I)}$$

$$b_2 = 0 \qquad \frac{1}{3}\frac{c_2}{a_2} = 0.1 \qquad \frac{1}{3}\frac{d_2}{a_2} = 0.1 \qquad \text{(II)}$$

$$b_2 = 0 \qquad \frac{1}{3}\frac{c_2}{a_2} = 0.1 \qquad \frac{1}{3}\frac{d_2}{a_2} = -0.1 \qquad \text{(III)}$$

which seem to be reasonable values for these constants if we assume that all the terms in a_2 are of the same order of magnitude as are the expressions c_2 and d_2.

It is thus seen that the types of the distribution formula in the case considered in this section, where the constants are fixed in accordance to MØLLER's considerations, are just the same as in the case where the constants are assumed to be independent of each other.

However, attention should be drawn to the fact that, if we assume the relations (61) to be valid, the lifetime of the mesons described by the vector wave-function is essentially the same as that of mesons given by the pseudoscalar wave-function so that an introduction of the pseudoscalar meson field would not remove the difficulty pointed out by Nordheim (see p. 6).

Summary.

A theory of β-disintegration, on the lines proposed by Yukawa, is developed in which the meson field is described by a four-vector and a pseudoscalar wave-function. In section 3, the general formula for the probability of β-decay of light elements is derived. In the following sections, it is shown that—in spite of the necessity of introducing several new universal constants—the general formula can, for allowed transitions, be reduced to one of two simple types, only. This result follows regardless of whether the new universal constants are considered to be independent of each other (section 4) or they are fixed in a way proposed by Møller (section 5).

A comparison of the theoretical distribution with the results of the experiments is difficult at the present time because the measurements on the β-spectra for allowed transitions and light elements are, except for ^{13}N, not sufficiently accurate for this purpose. Furthermore, the shape of the β-curve will be essentially changed if the nucleus formed after the β-process can be left in an excited state. Only in cases where the emitted γ-radiation is investigated, it becomes possible to build up the spectrum from its elementary components. The only element the

β-spectrum of which is measured with sufficient accuracy is ^{13}N, the results of the measurements on the γ-radiation emitted are, however, very divergent.

As regards the lifetime-energy connection, neither of the two types of disintegration probability is in disagreement with the experiments.

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